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PARTICIPATING LIFE INSURANCES IN AN EQUITY-LIBOR MARKET MODEL

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Participating life insurances in an equity-Libor market model

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Abstract : This article introduces an equity-Libor Market Model (LMM) that integrates the investment strategy into the valuation process of participating life insurances. Within this framework, we consider bond portfolios rebalanced across multiple maturities and provide a semi-analytical formula for approximating the fair value of liabilities. We then investigate the impact of the investment policy on the net asset value and the solvency capital requirement. To carry out this analysis, we propose a Monte Carlo method for generating sample paths under both Libor and real measures, alongside an estimation procedure under the real measure. The numerical illustration focuses on the asset-liability management of an endowment and a life annuity.

KEYWORDS : Libor market model, life insurance, asset-liability management

1 Introduction

The literature on the valuation of participating life insurance is vast, but few articles include bond investment strategies with multiple maturities. The bond portfolio is often a single maturity zero-coupon bond, with or without rebalancing, as in Barbarin and Devolder (2005) or Hainaut (2009 and 2010). In many models with stochastic interest rates, the portfolio of investments is instead modeled by a single asset, as in Bernard et al. (2005), Krayzler et al. (2016), or Hanna and Devolder (2023). To improve the analytical tractability of models, the interest rate is also often assumed constant. Without being exhaustive, we refer to Bacinello (2001), Ballotta et al. (2006), Gatzert and Kling (2007), or Gatzert (2008) for valuation models using this hypothesis.

In the literature about valuation with stochastic rates, the dominant approach is based on an affine process for the short-term rate, such as in the Hull and White model (1999). A competing framework is the Libor Market Model (LMM). It was introduced by Brace et al. (1997), and its features were explored by Jamshidian (1997) and Miltersen et al. (1997). The LMM has become a standard in the financial industry mainly due to its analytical tractability and its capacity to replicate the implied volatility surface of derivatives. Many extensions have been proposed since

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its creation. Andersen and Andreasen (2000), for instance, consider a version with local volatility. Andersen and Bortherton-Ratcliffe (2005) introduced stochastic volatility, whereas Eberlein et al. (2005) replaced the Brownian motions with general Lévy processes. Errais and Mercurio (2005) developed an extension with parameter uncertainty. We refer the reader to chapters 11 and 12 of Brigo and Mercurio (2007) for a survey of other variants of the LMM.

Nevertheless, few articles exploit the LMM for valuing life insurance liabilities. Among them, Schrager and Pelsser (2004) consider a variant of the hybrid equity-LMM to estimate participating insurance contracts. Gach et al. (2023) propose a mean-field Libor market model for the valuation of long-term guarantees. This approach may be regarded as a generic Libor market model with a time-dependent volatility structure, as introduced in Desmettre et al. (2021). It remedies the "blow-up" of interest rates during simulations. We refer to Gerhold (2011) for explanations about this phenomenon. The literature about hybrid equity-Libor market models is also scarce. We can cite Grzelak and Oosterlee (2012), who develop a hybrid LMM in which the equity part is ruled by a diffusion with stochastic volatility. Pilz and Schlogl (2015) propose a hybrid commodity and interest rate market model, constructed analogously to the multi-currency LMM. This is explained by the difficulty in specifying an analytically tractable model, as stock prices exhibit a different market dynamic than forward rates.

Our article contributes to the literature on the valuation of life insurance by integrating the bond-equity investment strategy in a hybrid LMM. We consider a with-profit contract for which the participation is linked to the performance of a portfolio of stocks and bonds with multiple maturities, regularly rebalanced. In this setting, we provide a semi-analytical formula for approximating liability values. We then discuss how the bond strategy impacts market risk indicators such as the net asset value (NAV) and solvency capital requirement (SCR). As most of capital indicators are based on simulations, we develop and test a Monte Carlo method for generating paths. Given the complexity of the drift in the dynamics of stock prices, we estimate the stock risk premium directly from sample paths. We also provide a calibration method for estimating the hybrid model under $\mathbb{P}$. The numerical analysis focuses on three bond strategies, including a cash-flow matching approach. This allows us to discuss the efficiency of reinvestment strategies based on constant duration, as imposed in Solvency II.

The outline of the article is as follows: We start by introducing the features of the insurance contract and the benchmark asset portfolio. Section 3 presents the hybrid equity-Libor market model. We discuss the valuation in Section 4. Section 5 details the Monte Carlo method for simulating sample paths under the spot Libor and real probability measure. The next section focuses on asset-liability management indicators. Section 8 develops an efficient estimation method under $\mathbb{P}$. In Section 8, we analyze the impact of the investment policy on the value and risk exposure of two insurance contracts. We also check the accuracy of the semi-analytical valuation formula.

2 The insurance contract and benchmark asset

We consider a participating insurance contract of maturity T_g, where $g \in \mathbb{N}$. The contract is purchased by a x-year old individual. The participation is linked to a benchmark portfolio of stocks and bonds whose market value is denoted by $(A_t)_{t\geq 0}$. The fair value of liabilities, also called the "best estimate", is denoted by $(L_t)_{t\geq 0}$. τ_x is the random remaining survival time of the individual. In case of premature decease, the capital is paid at the end of a period of length τ (a quarter, semester or year). The payment dates are $T_0, T_1, T_2, ..., T_g$ where $T_0 = \tau$ and $T_k = T_{k-1} + \tau$ for $k = 1, ..., g$.

The death and survival minimum benefits paid at time T_k are proportional to the reference asset and lower bounded by $d_k^{(min)}$ and $l_k^{(min)}$ capitalized at guaranteed rate $r_m \in \mathbb{R}^+$, for $k = 0,...,g$. We assume that the contract is financed by a lump sum payment and that benefits are positive. The guaranteed interest rate is noted $r_g \in \mathbb{R}^+$. The cash-flows in case of death or survival at time T_k are denoted by $CF_k^{(d)}$ and $CF_k^{(s)}$:

$$\begin{cases} CF_k^{(d)} = \max\left((1+r_g)^{T_k} d_k^{(min)}\,;\, \frac{A_{T_k}}{A_0} d_k^{(min)}\right) & T_{k-1} < \tau_x \leq T_k \quad k = 0, ..., g\,, \\ CF_k^{(s)} = \max\left((1+r_g)^{T_k} l_k^{(min)}\,;\, \frac{A_{T_k}}{A_0} l_k^{(min)}\right) & T_k < \tau_x\,. \end{cases}$$

In order to evaluate liabilities, we reformulate the cash-flows as the sum of minimum guaranteed benefits and a call option payoff:

$$\begin{cases} (1+r_g)^{T_k} d_k^{(min)} + \left(\frac{A_{T_k}}{A_0} d_k^{(min)} - d_k^{(min)} (1+r_g)^{T_k}\right)_+ & T_{k-1} < \tau_x \leq T_k \quad k = 0, ..., g\,, \\ (1+r_g)^{T_k} l_k^{(min)} + \left(\frac{A_{T_k}}{A_0} l_k^{(min)} - l_k^{(min)} (1+r_g)^{T_k}\right)_+ & T_k < \tau_x\,. \end{cases}$$

This framework is general enough to include a large range of products such as endowments, annuities or death insurances. For a participating endowment contract, we set $d_k^{(min)} = l_g^{(min)} \in \mathbb{R}^+$, for $k = 0, ..., g$. For a participating temporary life annuity, there is no death benefits $d_k^{(min)} = 0$, and the minimum (constant) survival benefits are equal to the discounted annuity: $l_k^{(min)} = l\,(1+r_g)^{-T_k}$. A temporary participating death insurance has no life benefits, $l_k = 0$ and $d_k = d\,(1+r_g)^{-T_k}$, with $d \in \mathbb{R}^+$.

All processes are defined on a complete probability space Ω, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a risk neutral probability measure, $\mathbb{Q}$. The cash account has a market value denoted $(B_t)_{t\geq 0}$ and we assume that $B_0 = 1$. The market is assumed to be arbitrage-free and therefore any discounted asset is a martingale under $\mathbb{Q}$. We denote the survival probability by ${}_{s-t}p_{x+t} = \mathbb{Q}(\tau_x > s \,|\, \tau_x \geq t)$ and the death probability by ${}_s q_{x+t} = \mathbb{Q}\left(\tau_x \leq t+s \,|\, \tau_x \geq t\right)$. If $s = 1$, we write q_{x+t} instead of ${}_1 q_{x+t}$. We define a function of time returning the date of the next payment,

$$\beta(t) = \min\left(i \,|\, t < T_i\,,\, i = 1, ...n\right).$$

To lighthen further developments, we denote by $V_L(t, T_j)$, the value of the discounted cash-flow paid at time T_j for an initial unit death or life benefit, i.e.

$$V_L(t, T_j) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{T_j}}\left((1+r_g)^{T_j} + \left(\frac{A_{T_j}}{A_0} - (1+r_g)^{T_j}\right)_+\right) |\mathcal{F}_t\right].$$

Under the assumption of independence between mortality and financial market, the fair value of liabilities at time $t \leq T_g$ is the sum of expected future discounted cash-flows, weighted by survival or death probabilities:

$$\begin{aligned} L_t \;=\; & \left({}_{T_{\beta(t)}-t}q_{x+t} d_{\beta(t)}^{(min)} + {}_{T_{\beta(t)}-t}p_{x+t} l_{\beta(t)}^{(min)}\right) V_L\left(t, T_{\beta(t)}\right) \\ & + \sum_{j=\beta(t)}^{g} \left({}_{T_j-t}p_{x+t}\, q_{x+T_j} d_{j+1}^{(min)} + {}_{T_{j+1}-t}p_{x+t} l_{j+1}^{(min)}\right) V_L\left(t, T_{j+1}\right). \end{aligned}$$

This fair-value is also called the “best estimate” in the Solvency II regulation. We assume that interest rates are stochastic and we denote the zero-coupon bond by $P(t,T) = \mathbb{E}^{\mathbb{Q}}\left(\frac{B_t}{B_T}|\mathcal{F}_t\right)$. The participating contract will be evaluated under an equivalent forward measure to $\mathbb{Q}$. The forward measure $\mathbb{F}(j)$ uses as numeraire, the zero-coupon bond of maturity T_j, $P(t, T_j)$. Under

$\mathbb{F}(j)$, all traded assets discounted by $P(t,T_j)$ are martingales. Using standard arguments, we rewrite the fair value of discounted unit benefit in terms of a forward expectation:

$$V_L(t,T_j) = P(t,T_j)(1+r_g)^{T_j} + \frac{1}{A_0}P(t,T_j)\mathbb{E}^{\mathbb{F}(j)}\left[\left(A_{T_j} - A_0(1+r_g)^{T_j}\right)_+ |\mathcal{F}_t\right]. \quad (1)$$

Valuing the best estimate of participating contracts is then equivalent to appraising call options on the benchmark asset. The complexity of this calculation mainly depends on the choice of the asset dynamics. In this work, interest rates are governed by the Libor Market Model (LMM) for two reasons. Firstly, this model is adopted by many insurance companies for evaluating their exposure to interest rates in Solvency II. Secondly, even though this model is defined in continuous time, it is based on a finite set of forward rates with discrete maturities. We will see that the advantage of such a specification is that it allows for the implementation of dynamic bond strategies for the asset. Within this framework, we assess the impact of bond allocation, and more generally of the ALM policy, on the value of participating insurance contracts.

Before detailing the LMM, we focus on the dynamic of the benchmark asset. We design it in order to replicate at best a realistic investment strategy. We consider a stock price process, $(S_t)_{t\geq 0}$ and n zero-coupon bonds with maturities $\{T_0, T_1, T_2, \ldots, T_n\}$ where $T_0 = \tau$ and $T_k = T_{k-1} + \tau$ for $k = 1, ..., n$. The longest bond maturity is at least equal to the contract expiry date, $T_n \geq T_g$ and by construction, intermediate bond maturities coincide with payment dates of the insurance policy.

We adopt an hybrid continuous and discrete time rebalancing strategy for the asset. We assume that the insurer invests a fraction $\pi \in [0,1]$ of the total asset in stocks. The remaining is invested in bonds. The structure of the bond portfolio is stored in a positive matrix $\eta_u^i \in \mathbb{R}^+$ for $i = 1, ..., g$ and $u = 0, ..., n$. More precisely, we assume that at time $t \in [T_{i-1}, T_i)$ with $i \leq g$, the insurer holds an amount

$$(1-\pi)\frac{\eta_u^i}{\sum_{v=i}^n \eta_v^i} A_t = (1-\pi)\frac{\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^n \eta_v^{\beta(t)}} A_t \quad (2)$$

of bond $P(t,T_u)$, of maturity T_u for $u = \beta(t), ..., n$. This expression is flexible enough to implement various type of bond strategies. In Equation (2), we use $\sum_{v=i}^n \eta_v^i$ as normalization factor to avoid to introduce the constraint $\sum_{v=i}^n \eta_v^i = 1$. In the numerical illustration, we focus on three investment policies. In the first one, the insurer homogenously invests $(1-\pi)A_t$ in bonds with expiry dates between T_{m_1} and T_{m_2}, with $m_1, m_2 \in \mathbb{N}$, $m_1 \leq g \leq m_2 \leq n$:

$$\eta_u^i = \begin{cases} 1 & m_1 \leq u \leq m_2 \\ 0 & otherwise \end{cases}, \; i = 1, ..., g. \quad (3)$$

This allocation is called "fixed maturity bucket (FMB) strategy".

In the second strategy, the bond maturities are rebalanced at dates T_i to approximatively keep constant the duration of the portfolio. At time T_i, $(1-\pi)A_t$ is invested into bonds of durations from $T_{i+m_1} - T_i$ to $T_{i+m_2} - T_i$, with $m_1, m_2 \in \mathbb{N}$, $i + m_1 \leq i + m_2 \leq n$:

$$\eta_u^i = \begin{cases} 1 & m_1 \leq u - i \leq m_2 \\ 0 & otherwise \end{cases}, \; i = 1, ..., g. \quad (4)$$

This policy of investment is called "constant duration bucket (CDB) strategy". This is the only strategy accepted by the regulator for computing best estimates in Solvency II. The motivation for such a choice is that insurance companies must simulate their investment on an ongoing basis even if liabilities cash-flows are in run-off. In the numerical illustration, we will quantify the impact of this rule on the value of insurance contracts.

The last investment policy is called "partial cash-flow matching (PCFM) strategy". We allocate $(1-\pi)\,A_t$ in a portfolio of bonds with maturities corresponding to liability payment dates. Furthermore, the invested amounts are proportional to expected minimum benefits, discounted with the initial yield curve:

$$\eta_u^i = \begin{cases} e^{r_g T_u}\left({}_{T_{u-1}-T_{i-1}}p_{x+T_{i-1}}\, q_{x+T_{u-1}} d_u^{(min)} + {}_{T_u-T_{i-1}}p_{x+T_{i-1}} l_u^{(min)}\right)\frac{P(0,T_u)}{P(0,T_{i-1})} & i \leq u \leq g\,, \\ 0 & otherwise\,. \end{cases} \tag{5}$$

In a first stage, we will quantify the impact of the investment strategy of the reference asset on the fair value of the participating policy. But before discussing the valuation method, we need to specify the market dynamics.

3 The hybrid equity-Libor market model

In addition to the stock price process, $(S_t)_{t\geq 0}$, we consider n forward rates. The forward rate at time t, for a future operation with fixing at time T_{i-1} and settled at T_i (with $t \leq T_{i-1} \leq T_i$) is denoted by

$$F_t^i = F(t, T_{i-1}, T_i) \qquad i = 1, \ldots, n\,.$$

Fixing dates are equispaced by τ, and correspond to liability and bond maturities. In the Libor Market Model (LMM), the short term rate is not explicitely defined. We instead specify the dynamic of forward rates under their respective equivalent measures. For this reason, we do not specify the dynamic of $(S_t)_{t\geq 0}$ under the risk neutral measure. We instead focus on the forward stock price, of maturity T_n, defined as follows

$$Y_t^n \;=\; \frac{S_t}{P(t,T_n)} \quad , \quad t \leq T_n.$$

By definition, Y_t^n is a martingale under the forward measure $\mathbb{F}(T_n)$, with $P(t,T_n)$ as numeraire: $\mathbb{E}^{\mathbb{F}(n)}\left(Y_s^n|\mathcal{F}_t\right) = Y_t^n$ and $\lim_{t\to T_n} Y_t^n = S_{T_n}$. By construction, the current value of the stock price is the discounted expected forward price:

$$S_t = P(t,T_n)\mathbb{E}^{\mathbb{F}(n)}\left(Y_{T_n}^n \,|\, \mathcal{F}_t\right)\,.$$

In the LMM, forward rates $\left(F_t^i\right)_{0\leq t\leq T_{i-1}}$ are ruled by the following SDE under measure $\mathbb{F}(i)$,

$$dF_t^i \;=\; \sigma_i(t)\,\left(F_t^i+\alpha\right)\,\Sigma_{i,:}\,d\boldsymbol{W}_t^{\mathbb{F}(i)} \quad i = 0,...,n\,,\, t \leq T_i\,.$$

where $\alpha \in \mathbb{R}^+$ and $\boldsymbol{W}_t^{\mathbb{F}(i)} = \left(W_t^{\mathbb{F}(i),1},...,W_t^{\mathbb{F}(i),p}\right)^\top$ is a vector of independent Brownian motions of dimension $p \leq n$. $\Sigma_{i,:}$ is the i^{th} line of a $n\times p$ matrix, denoted by Σ while $\sigma_i(t)$ is a integrable function of time. As Y_t^n is a martingale under the forward measure $\mathbb{F}^{(n)}$, we model it by the following Doleans Dade exponential for $t \leq T_n$:

$$\frac{dY_t^n}{Y_t^n} \;=\; \sigma_S\left(\Sigma_{YF}d\boldsymbol{W}_t^{\mathbb{F}(n)} + \Sigma_{YY}dW_t^{S,\mathbb{F}(n)}\right)\, t \leq T_n$$

where Σ_{YF} is a line vector of dimension p and Σ_{YY} is a scalar. $W_t^{S,\mathbb{F}(n)}$ is a Brownian motion, independent from $\boldsymbol{W}_t^{\mathbb{F}(n)}$. The $(n+1)\times(p+1)$-matrix $\begin{pmatrix} \Sigma & 0 \\ \Sigma_{YF} & \Sigma_{YY} \end{pmatrix}$ is the lower Cholesky decomposition of the correlation matrix between forward rates and the forward stock price. If $\rho_{F^iF^j}$ and ρ_{YF^j} are respectively the correlations between F_t^i, F_t^j and F_t^j, Y_t , we have that:

$$\left(\begin{array}{cc}\Sigma & 0\\ \Sigma_{YF} & \Sigma_{YY}\end{array}\right)\left(\begin{array}{cc}\Sigma^\top & \Sigma_{YF}^\top\\ 0 & \Sigma_{YY}\end{array}\right) = \left(\begin{array}{cccc}1 & \dots & \rho_{F^nF^1} & \rho_{YF^1}\\ \vdots & \ddots & \vdots & \vdots\\ \rho_{F^nF^1} & \dots & 1 & \rho_{YF^n}\\ \rho_{YF^1} & \dots & \rho_{YF^n} & 1\end{array}\right).$$

Note that our framework differs from that of Schrager and Pelsser (2004). Firstly, forward bond prices are driven by geometric diffusion instead of forward rates. Secondly, the reference asset for the participation is a single risky fund instead of a portfolio of bonds and stocks. Our approach also slightly differs from Grzelak and Oosterlee (2012), who define the dynamics of stocks under the risk-neutral measure. Instead, we define it immediately under the forward measure, which slightly reduces the complexity of the valuation. Applying the Itô's lemma to log-prices allows us to show that $Y_{T_n}^n$ is log-normal under the measure $\mathbb{F}(n)$:

$$Y_{T_n}^n \sim N\left(-\frac{1}{2}\sigma_{Y,n}^2(t)\,;\,\sigma_{Y,n}^2(t)\right),$$

where $\sigma_{Y,n}^2(t)$ is equal to

$$\sigma_{Y,n}^2(t) = \sigma_S^2(T_n - t)\,.$$

To specify the stock price under other forward measures than $\mathbb{F}(n)$, we need to infer the dynamics of bond prices. The condition of absence of arbitrage implies that for $t \leq T_k$, the bond price of maturity T_k is equal to the a product of forward discount bonds:

$$P(t,T_k) = P(t,T_{\beta(t)})\prod_{j=\beta(t)+1}^{k}\frac{1}{1+\tau F_t^j}\,.$$

In practice (e.g. in simulations), $P(t,T_{\beta(t)})$ is unknown and we need to use the approximation $L(t,T_{\beta(t)}) \approx F(T_{\beta(t)-1},T_{\beta(t)-1},T_{\beta(t)})$ for $t \in [T_{\beta(t-1)},T_{\beta(t)})$. Using this, the zero-coupon bond is approached by

$$P(t,T_k) \approx \frac{1}{1+\left(T_{\beta(t)}-t\right)F_{T_{\beta(t)-1}}^{\beta(t)}}\prod_{j=\beta(t)+1}^{k}\frac{1}{1+\tau F_t^j}\,. \tag{6}$$

Based on this last statement, we infer in the next proposition the SDE driving $P(t,T_k)$. In most of the developments, the time-dependent part in SDE's is unnecessary and complex. For this reason, We adopt the notation (...) for terms that are not needed in the following developments.

Proposition 1. *For $i \leq k$, the zero-coupon bond price of maturity T_k is ruled by the following SDE under $\mathbb{F}^{(i)}$*

$$dP(t,T_k) = (\dots)dt - P(t,T_k)\sum_{j=\beta(t)+1}^{k}\frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j}\,d\boldsymbol{W}_t^{\mathbb{F}(i)} \tag{7}$$

Proof. From standard stochastic calculus, the differential of the product is:

$$d\left(\prod_{j=\beta(t)+1}^{k}\frac{1}{1+\tau F_t^j}\right) = -\sum_{j=\beta(t)+1}^{k}\frac{1}{1+\tau F_t^j}\left(\prod_{j=\beta(t)+1}^{k}\frac{1}{1+\tau F_t^j}\right)dF_t^j + (\dots)\,dt\,.$$

From Equation (6), we infer the differential of the ZC bond:

$$
\begin{aligned}
dP(t,T_k) &= (...)dt - P(t,T_k) \sum_{j=\beta(t)+1}^{k} \frac{1}{1+\tau F_t^j} dF_t^j \\
&= (...)dt - P(t,T_k) \sum_{j=\beta(t)+1}^{k} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(j)}
\end{aligned}
$$

We switch to the measure $\mathbb{F}(i)$ with numeraire $P(t,T_i)$ (such that $T_i \leq T_k$). From Proposition 6 in appendix A, we have that

$$
d\boldsymbol{W}_t^{\mathbb{F}(j)} = d\boldsymbol{W}_t^{\mathbb{F}(j+1)} - \frac{\tau\sigma_{j+1}(t)\left(F_t^{j+1}+\alpha\right)\Sigma_{j+1,:}^{\top}}{1+\tau F_t^{j+1}} dt\,.
$$

Plugging this in the expression of $dP(t,T_k)$ leads to

$$
dP(t,T_k) = (...)\,dt - P(t,T_k) \sum_{j=\beta(t)+1}^{k} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)}\,.
$$

□

In order to determine the dynamic of the total asset in the next section, we need the following proposition that provides the dynamic of stocks under different forward measures from $\mathbb{F}(n)$.

Proposition 2. *Under the measure $\mathbb{F}(i)$ with $i \geq \beta(t)+1$, the stock price is solution of the SDE:*

$$
\begin{aligned}
dS_t &= S_t \mu_S^{\mathbb{F}(i)}(t)\,dt - S_t \sum_{j=\beta(t)+1}^{n} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)} \\
&\quad + S_t\sigma_S\left(\Sigma_{YF} d\boldsymbol{W}_t^{\mathbb{F}(i)} + \Sigma_{YY} dW_t^{S,\mathbb{F}(i)}\right)\,.
\end{aligned} \tag{8}
$$

where $\mu_S^{\mathbb{F}(i)}(t)$ is a $\mathcal{F}_t$-adapted process without a closed-form expression.

Proof. By definition, $S_t = Y_t^n\, P(t,T_n)$. Using Eq. (7), the differential of S_t is therefore

$$
\begin{aligned}
dS_t &= Y_t^n\, dP(t,T_n) + P(t,T_n)\, dY_t^n + d\left\langle Y_t^n\,,\, P(t,T_n)\right\rangle \\
&= S_t(...)\,dt - S_t \sum_{j=\beta(t)+1}^{n} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(n)} \\
&\quad + S_t\sigma_S\left(\Sigma_{YF} d\boldsymbol{W}_t^{\mathbb{F}(n)} + \Sigma_{YY} dW_t^{S,\mathbb{F}(n)}\right)
\end{aligned}
$$

Operating again a change of mesure from $\mathbb{F}(n)$ to $\mathbb{F}(i)$, we obtain the result. □

4 Valuation of liabilities

The valuation of the participating option in Equation (1), requires determining the dynamic of the benchmark portfolio, A_t, under a forward measure. Let us recall that we invest a fraction π of A_t in equity and $(1-\pi)A_t$ into bonds. The bond strategy is defined by the matrix η_u^i for $i = 1, ..., g$ and $u = 0, ..., n$, according to Equation (2).

Proposition 3. *Under the forward measure $\mathbb{F}(i)$, with $i \geq \beta(t)+1$,the benchmark asset discounted by $P(t,T_i)$ is ruled by the following SDE:*

$$
\begin{aligned}
d\left(\frac{A_t}{P(t,T_i)}\right) &= \frac{A_t}{P(t,T_i)} \sum_{j=\beta(t)+1}^{i} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)} \qquad (9)\\
&+\frac{A_t}{P(t,T_i)}\pi\sigma_S\left(\Sigma_{YF} d\boldsymbol{W}_t^{\mathbb{F}(i)} + \Sigma_{YY} dW_t^{S,\mathbb{F}(i)}\right)\\
&-\frac{A_t\pi}{P(t,T_i)} \sum_{j=\beta(t)+1}^{n} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)}\\
&-\frac{A_t(1-\pi)}{P(t,T_i)} \sum_{j=\beta(t)+1}^{n} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} \frac{\sum_{u=j}^{n}\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}} d\boldsymbol{W}_t^{\mathbb{F}(i)}.
\end{aligned}
$$

Proof. Under $\mathbb{F}(i)$, the discounted portfolio is a martingale. On the other hand, we have that

$$
d\left(\frac{A_t}{P(t,T_i)}\right) = \frac{dA_t}{P(t,T_i)} + A_t d\left(\frac{1}{P(t,T_i)}\right) + \frac{1}{2} d\left\langle A_t, \frac{1}{P(t,T_i)}\right\rangle,
$$

where by definition of the benchmark portfolio,

$$
\begin{aligned}
\frac{dA_t}{P(t,T_i)} &= \frac{A_t}{P(t,T_i)}\pi\frac{dS_t}{S_t} + \frac{(1-\pi)\eta_{\beta(t)}^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}} \frac{A_t}{P(t,T_i)} \frac{dP(t,T_{\beta(t)})}{P(t,T_{\beta(t)})}\\
&+ \sum_{u=\beta(t)+1}^{n} \frac{(1-\pi)\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}} \frac{A_t}{P(t,T_i)} \frac{dP(t,T_u)}{P(t,T_u)}.
\end{aligned}
$$

Furthermore, applying the Itô's lemma to Equation (6) leads to

$$
d\left(\frac{1}{P(t,T_i)}\right) = (...)dt - \frac{1}{P(t,T_i)^2} dP(t,T_i).
$$

In the LMM, we assume that the yield of $P(t,T_{\beta(t)})$ is constant for $t \in \left[T_{\beta(t)-1}, T_{\beta(t)}\right]$ and therefore

$$
\frac{dP(t,T_{\beta(t)})}{P(t,T_{\beta(t)})} = F_{T_{\beta(t)-1}}^{\beta(t)} dt
$$

We finally infer the following asset dynamic:

$$
\begin{aligned}
d\left(\frac{A_t}{P(t,T_i)}\right) &= (...)\,dt - \frac{A_t}{P(t,T_i)} \frac{dP(t,T_i)}{P(t,T_i)}\\
&+\frac{A_t}{P(t,T_i)}\pi\frac{dS_t}{S_t} + \frac{(1-\pi)\eta_{\beta(t)}^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}} \frac{A_t}{P(t,T_i)} F_{T_{\beta(t)-1}}^{\beta(t)} dt\\
&+ \sum_{u=\beta(t)+1}^{n} \frac{(1-\pi)\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}} \frac{A_t}{P(t,T_i)} \frac{dP(t,T_u)}{P(t,T_u)}.
\end{aligned}
$$

We next substitute $\frac{dS_t}{S_t}$ by Eq. (8) and combine previous relations to obtain the SDE driving

$\frac{A_t}{P(t,T_i)}$. Under $\mathbb{F}^{(i)}$, $\frac{A_t}{P(t,T_i)}$ is a martingale, the drift is therefore null and we find that

$$d\left(\frac{A_t}{P(t,T_i)}\right) = \frac{A_t}{P(t,T_i)} \sum_{j=\beta(t)+1}^{i} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)}$$
$$+\frac{A_t}{P(t,T_i)}\pi\sigma_S\left(\Sigma_{YF} d\boldsymbol{W}_t^{\mathbb{F}(i)} + \Sigma_{YY} dW_t^{S,\mathbb{F}(i)}\right)$$
$$-\frac{A_t}{P(t,T_i)}\pi \sum_{j=\beta(t)+1}^{n} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)}$$
$$-\frac{A_t}{P(t,T_i)} \sum_{u=\beta(t)+1}^{n} \frac{(1-\pi)\,\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}} \sum_{j=\beta(t)+1}^{u} \frac{\sigma_j(t)\left(F_t^j+\alpha\right)\Sigma_{j,:}}{1+\tau F_t^j} d\boldsymbol{W}_t^{\mathbb{F}(i)}.$$

We conclude by switching the summation order in the last equation:

$$\sum_{u=\beta(t)+1}^{n}\sum_{j=\beta(t)+1}^{u} \ldots = \sum_{j=\beta(t)+1}^{n}\sum_{u=j}^{n} \ldots$$

□

Equation (9) reveals that $\frac{A_t}{P(t,T_i)}$ is a geometric diffusion. Its statistical distribution is nevertheless unknown since it depends on forward rates. A common and robust assumption consists to "freeze" the ratios $\frac{(F_0^j+\alpha)}{1+\tau F_0^j}$ to their initial value as e.g. recommanded in Brigo and Mercurio (2007).

Conjecture 1. *For j=1,...,n We assume that*

$$\psi_j = \frac{\left(F_0^j+\alpha\right)}{1+\tau F_0^j} \approx \frac{\left(F_t^j+\alpha\right)}{1+\tau F_t^j}. \tag{10}$$

Using this assumption, the dynamic of $\frac{A_t}{P(t,T_i)}$ is rewritten as follows:

$$d\left(\frac{A_t}{P(t,T_i)}\right) = \frac{A_t}{P(t,T_i)}\pi\sigma_S\Sigma_{YY} dW_t^{S,\mathbb{F}(i)} + \frac{A_t}{P(t,T_i)}$$
$$\times\left(\pi\,\sigma_S\Sigma_{YF} + \sum_{j=\beta(t)+1}^{i}\sigma_j(t)\psi_j\Sigma_{j,:}\right.$$
$$\left. - \sum_{j=\beta(t)+1}^{n}\sigma_j(t)\psi_j\Sigma_{j,:}\left(\pi + (1-\pi)\frac{\sum_{u=j}^{n}\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}}\right)\right) d\boldsymbol{W}_t^{\mathbb{F}(g)}.$$

As all the coefficients are constant or deterministic, we infer that $\frac{A_t}{P(t,T_i)}$ is log-normal under the forward measure $\mathbb{F}(i)$. Let us define $\sigma_{A,i}^2(t,s)$, the variance of $A_s|\mathcal{F}_t$ under the measure $\mathbb{F}(i)$:

$$\sigma_{A,i}^2(t,s) := \pi^2\sigma_S^2\left(\Sigma_{YY}\right)^2(s-t) + \int_t^s\left(\pi\,\sigma_S\Sigma_{YF} + \sum_{j=\beta(z)+1}^{i}\sigma_j(z)\psi_j\Sigma_{j,:}\right. \tag{11}$$
$$\left. - \sum_{j=\beta(z)+1}^{n}\sigma_j(z)\psi_j\Sigma_{j,:}\left(\pi + (1-\pi)\frac{\sum_{u=j}^{n}\eta_u^{\beta(z)}}{\sum_{v=\beta(z)}^{n}\eta_v^{\beta(z)}}\right)\right)^{2\top} dz\,,$$

where we adopt the notation $\boldsymbol{x}^{2\top} = \boldsymbol{x}\boldsymbol{x}^\top$ to shorten expressions. In the numerical illustration, the integral in Equation (11) is computed numerically. Applying the Itô's lemma to $\ln \frac{A_t}{P(t,T_i)}$, allows us to infer that:

$$\ln\left(\frac{A_{T_i}}{P(T_i,T_i)} \Big/ \frac{A_t}{P(t,T_i)}\right) \sim N\left(-\frac{1}{2}\sigma^2_{A,i}(t,T_i)\,;\,\sigma^2_{A,g}(t,T_i)\right).$$

As we are in a log-normal framework, we can easily retrieve a "Black & Scholes" like equation for valuing the participating contract.

Proposition 4. *Let $\Phi(.)$ be the cumulative distribution function (cdf) of a $N(0,1)$ random variable. For $j = 0,...,g$, let us define*

$$\begin{aligned} d_2^j(t) &= \frac{\ln\left(\frac{P(t,T_j)A_0(1+r_g)^{T_j}}{A_t}\right) + \frac{1}{2}\sigma^2_{A,j}(t,T_j)}{\sigma_{A,j}(t,T_j)}, \\ d_1^j(t) &= d_2(t) - \sigma_{A,j}(t,T_j)\,. \end{aligned}$$

Under the hypothesis (10), the best estimate of the participating contract is approximated by

$$\begin{aligned} L_t &= \left({}_{T_{\beta(t)}-t}q_{x+t}d^{(min)}_{\beta(t)} + {}_{T_{\beta(t)}-t}p_{x+t}l^{(min)}_{\beta(t)}\right) V_L\left(t, T_{\beta(t)}\right) \\ &\quad + \sum_{j=\beta(t)}^{g} \left({}_{T_j-t}p_{x+t}\, q_{x+T_j} d^{(min)}_{j+1} + {}_{T_{j+1}-t}p_{x+t}l^{(min)}_{j+1}\right) V_L\left(t, T_{j+1}\right). \end{aligned} \tag{12}$$

where $V_L(t,T_j)$ is the sum of a discounted guaranteed benefit and of the participating option:

$$V_L(t,T_j) = P(t,T_j)\,(1+r_g)^{T_j} + \frac{A_t}{A_0}\,\Phi\left(-d_1^j(t)\right) - P(t,T_j)\,(1+r_g)^{T_j}\,\Phi\left(-d_2^j(t)\right). \tag{13}$$

The accuracy of this approximation will be checked in the numerical analysis by comparison with prices obtained by Monte-Carlo simulations.

5 Asset simulations

Before detailing the simulation method, we need to choose a measure under which we will perform them. In this article, we choose the spot Libor measure at time t, noted $\mathbb{L}(t)$. This is the shortest term forward measure that is still in force at time t:

$$\mathbb{L}(t) = \mathbb{F}(\beta(t)+1)\,.$$

We can prove (see e.g. Brigo and Mercurio 2007) that it is the equivalent of the risk neutral measure but with a numeraire that is a cash account, $(B_{T_k})_{k=0,...,n}$ capitalized at Libor rate:

$$B_{T_k} = B_{T_0}\prod_{i=1}^{k}(1+\tau L(T_{i-1},T_i)) = B_{T_0}\prod_{i=1}^{k}(1+\tau F(T_{i-1},T_{i-1},T_i))\,.$$

We alsoneed this measure to construct later the dynamics of stocks and forward rates under the real measure $\mathbb{P}$. This step is required for computing the solvency capital according to Solvency II. We will come back on this point in Section 6.

Proposition 5. *The dynamic of the forward stock price, Y_t^n, under the spot Libor measure $\mathbb{L}(t)$ is given by*

$$\begin{aligned} \frac{dY_t^n}{Y_t^n} &= \sum_{k=\beta(t)+1}^{n} \frac{\tau\sigma_k(t)\sigma_S\,\left(F_t^k+\alpha\right)\,\rho_{YF^k}}{1+\tau F_t^k}dt \\ &\quad +\sigma_S\left(\Sigma_{YF}d\boldsymbol{W}_t^{\mathbb{L}(t)} + \Sigma_{YY}dW_t^{S,\mathbb{L}(t)}\right). \end{aligned} \tag{14}$$

Whereas forward rates F_t^j with $j \geq \beta(t)+1$ under $\mathbb{L}(t)$ are ruled by

$$\frac{dF_t^j}{\left(F_t^j+\alpha\right)} = \sum_{k=\beta(t)+1}^{n} \frac{\tau\sigma_k(t)\sigma_j(t)\rho_{j,k}\left(F_t^k+\alpha\right)}{1+\tau F_t^k}dt + \sigma_j(t)\,\Sigma_{j,:}\,d\boldsymbol{W}_t^{\mathbb{L}(t)} \tag{15}$$

Proof. From Prop. 6 in Appendix A, if $\boldsymbol{\theta}_t^i = -\frac{\tau\sigma_i(t)\left(F_t^i+\alpha\right)\Sigma_{i,:}^\top}{1+\tau F_t^i}$, we have that

$$d\boldsymbol{W}_t^{\mathbb{F}(i-1)} = d\boldsymbol{W}_t^{\mathbb{F}(i)} + \boldsymbol{\theta}_t^i dt\,,$$

and $dW_t^{S,\mathbb{F}(n-1)} = dW_t^{S,\mathbb{F}(n)}$ (because of the independence between $W_t^{S,\mathbb{F}(n)}$ and $\boldsymbol{W}_t^{\mathbb{F}(n)}$). As $\Sigma_{n,:}\Sigma_{YF}^\top = \rho_{YF^n}$, we find that

$$\begin{aligned}\frac{dY_t^n}{Y_t^n} &= \frac{\tau\sigma_n(t)\sigma_S\,\left(F_t^n+\alpha\right)\,\rho_{YF^n}}{1+\tau F_t^n}dt \\ &\quad +\sigma_S\left(\Sigma_{YF}d\boldsymbol{W}_t^{\mathbb{F}(n-1)}+\Sigma_{YY}dW_t^{S,\mathbb{F}(n-1)}\right)\,.\end{aligned}$$

We infer Equation (14) by iterating. Statement (15) is a direct consequence of the same Proposition 6. □

Simulations of Y_t^n and F_t^j's are performed by Euler discretization of Equations (14) and (15). If the discretization step is denoted by $\Delta \in \mathbb{R}^+$, $Y_{t+\Delta}^n$ is simulated from Y_t^n using the recursion:

$$\begin{aligned}\frac{Y_{t+\Delta}^n - Y_t^n}{Y_t^n} &= \sum_{k=\beta(t)+1}^{n}\frac{\tau\sigma_k(t)\sigma_S\,\left(F_t^k+\alpha\right)\,\rho_{YF^k}}{1+\tau F_t^k}\Delta \\ &\quad +\sigma_S\left(\Sigma_{YF}\Delta\boldsymbol{W}_t^{\mathbb{L}(t)}+\Sigma_{YY}\Delta W_t^{S,\mathbb{L}(t)}\right)\,,\end{aligned} \tag{16}$$

where $\Delta\boldsymbol{W}_t^{\mathbb{L}(t)}$ and $\Delta W_t^{S,\mathbb{L}(t)}$ are respectively realizations of normal random variables of sizes p and 1, with null means and standard deviations $\sqrt{\Delta}$. In a similar manner, the forward rates are built with the following formula:

$$\frac{F_{t+\Delta}^j - F_t^j}{\left(F_t^j+\alpha\right)} = \sum_{k=\beta(t)+1}^{j}\frac{\tau\sigma_k(t)\sigma_j(t)\rho_{j,k}\left(F_t^k+\alpha\right)}{1+\tau F_t^k}\Delta + \sigma_j(t)\,\Sigma_{j,:}\,\Delta\boldsymbol{W}_t^{\mathbb{L}(t)}, \tag{17}$$

for $j=1,...,n$. After simulating forward rates and stock prices, we can reconstruct the term structure of bond prices and determine the stock value using the relations:

$$\begin{cases} P(t,T_k) & \approx \frac{1}{1+\left(T_{\beta(t)}-t\right)F_{T_{\beta(t)-1}}^{\beta(t)}}\prod_{j=\beta(t)+1}^{k}\frac{1}{1+\tau F_t^j}\,, \\ S_t & = P(t,T_n)Y_t^n\,. \end{cases}$$

Using again an Euler discretization, we calculate the sample path of the benchmark portfolio:

$$\begin{aligned}\frac{A_{t+\Delta}^n - A_t^n}{A_t^n} &= \pi\frac{S_{t+\Delta}^n - S_t^n}{S_t^n} + (1-\pi)\times \\ &\sum_{u=\beta(t)}^{n}\frac{\eta_u^{\beta(t)}}{\sum_{v=\beta(t)}^{n}\eta_v^{\beta(t)}}\frac{P(t+\Delta,T_u)-P(t,T_u)}{P(t,T_u)}\,.\end{aligned} \tag{18}$$

We will use this procedure to evaluate participating contracts by simulations and estimate the accuracy of the approximated formula (12).

The computation of risk management indicators introduced in Section 6, requires simulations under the real measure $\mathbb{P}$. From the Radon-Nykodym theorem, the dynamics of the forward stock price and rates under $\mathbb{P}$ are obtained by substituting

$$\begin{cases} \boldsymbol{W}_t^{\mathbb{L}(t)} & = \boldsymbol{W}_t^{\mathbb{P}} + \boldsymbol{\theta}_t dt \,, \\ W_t^{S,\mathbb{L}(t)} & = W_t^{S,\mathbb{P}} + \theta_t^S dt \,, \end{cases}$$

where $\boldsymbol{W}_t^{\mathbb{P}}$, $W_t^{S,\mathbb{P}}$ are $\mathbb{P}$-Brownian motions and $\boldsymbol{\theta}_t$, θ_t^S are risk premiums. $\boldsymbol{\theta}_t$, θ_t^S are $\mathcal{F}_t-$adapted processes. Knowing these risk premiums, we can adapt Equations (16), (17), (18) to simulate sample paths under $\mathbb{P}$.

In practice, estimating the risk premiums for the interest rate risk is challenging. In the numerical illustration, we adopt a conservative assumption which is $\boldsymbol{W}_t^{\mathbb{L}(t)} = \boldsymbol{W}_t^{\mathbb{P}}$. This assumption is commonly applied by practitioners. On the other hand, we choose θ_t^S such that the average stock return under $\mathbb{P}$ denoted by $\mu \in \mathbb{R}^+$, is constant. From Proposition 2, we infer the dynamic of the stock price under the real measure:

$$\begin{aligned} dS_t &= \mu S_t dt - S_t \sum_{k=\beta(t)+1}^{n} \frac{\sigma_k(t)\left(F_t^k+\alpha\right)\,\Sigma_{k,:}}{1+\tau F_t^k}\, d\boldsymbol{W}_t^{\mathbb{L}(t)} \\ &\quad + S_t \sigma_S \left(\Sigma_{YF} d\boldsymbol{W}_t^{\mathbb{P}} + \Sigma_{YY} dW_t^{S,\mathbb{P}}\right) . \end{aligned} \tag{19}$$

and that the risk premium must therefore be equal to

$$\theta_t^S = \frac{\mu - \mu_S^{\mathbb{L}}(t)}{\sigma_S \Sigma_{YY}} \,,$$

where $\mu_S^{\mathbb{L}}(t)$ is the drift of stock prices, as defined in Proposition 2, under the Libor measure. Nevertheless, the analytical expression of $\mu_S^{\mathbb{L}}(t)$ is too complex for being determined. We instead estimate it directly from stock and forward stock prices, simulated under the Libor measure $\mathbb{L}(t)$. To do this, we combine Euler discretization of Equations (19), (14) and infer from simulated variations of stock and forward stock prices that

$$\begin{aligned} \mu_S^{\mathbb{L}}(t)\,\Delta = & \frac{S_{t+\Delta}^n - S_t^n}{S_t^n} - \frac{Y_{t+\Delta}^n - Y_t^n}{Y_t^n} + \\ & \sum_{k=\beta(t)+1}^{n} \frac{\sigma_k(t)\left(F_t^k+\alpha\right)}{1+\tau F_t^k}\left(\tau\sigma_S\rho_{SF^k}\Delta + \Sigma_{k,:}\Delta\boldsymbol{W}_t^{\mathbb{L}(t)}\right) . \end{aligned}$$

Let us denote by $\tilde{Y}_t$ and $\tilde{S}_t$, simulated sample paths of forward stock and stock prices under the real measure, $\mathbb{P}$. As $\boldsymbol{W}_t^{\mathbb{L}(t)} = \boldsymbol{W}_t^{\mathbb{P}}$ and

$$\Delta W_t^{S,\mathbb{L}(t)} = \Delta W_t^{S,\mathbb{P}} + \frac{\mu - \mu_S^{\mathbb{L}}(t)}{\sigma_S \Sigma_{YY}}\Delta,$$

we obtain from Equation (16) that

$$\begin{cases} \frac{\tilde{Y}_{t+\Delta}^n - \tilde{Y}_t^n}{\tilde{Y}_t^n} & = \frac{Y_{t+\Delta}^n - Y_t^n}{Y_t^n} + \left(\mu - \mu_S^{\mathbb{L}}(t)\right)\Delta \,, \\ \tilde{S}_t & = P(t,T_n)\tilde{Y}_t^n \,. \end{cases} \tag{20}$$

Using this adjustment allows to simulate sample path of stock price under the real measure $\mathbb{P}$. As explained in the following section, we will combine asset simulations with the approximated valuation formula of liabilities to measure the exposure to market risk.

6 Asset-Liability management

The investment policy impacts financial performance and exposure to market risk. For an insurer, it is a matter of crucial importance to adapt this policy to their risk appetite. And for a given exposure to risk, the insurer has to select the most performing investment strategy. In this section, we define the indicators of risk exposure and global performance, compliant with the Solvency II regulation.

The benchmark portfolio, $(A_t)_{t\geq 0}$, reflects the performance of insurer's investments over time. We denote by $(I_t)_{t\geq 0}$, the insurer's total asset[1]. We assume that the insurer's investment policy is strictly the same as that of the benchmark portfolio. If we remember that $CF_k^{(d)}$ and $CF_k^{(s)}$ are the cash-flows paid at times T_k for $k = 0, ..., g$, the insurer's total asset at time t is linked to cash-flows by:

$$I_t = I_0 \frac{A_t}{A_0} - \sum_{k=0}^{\beta(t)-1} \left(\mathbf{1}_{\{T_{k-1} < \tau_x \leq T_k\}} CF_k^{(d)} + \mathbf{1}_{\{T_k < \tau_x\}} CF_k^{(s)} \right) \frac{A_t}{A_{T_k}} . \tag{21}$$

Using this relation, we simulate sample paths of the insurer's asset under $\mathbb{P}$, with the procedure described in Section 5. In each of these scenarios, we approximate the best estimate, L_t, using the closed-form formula (12). Without this formula, we would need to perform nested simulations or to implement a least square Monte-Carlo (LSMC) method. Nested simulations are computationally intensive and avoided in practice for this reason. The LSMC method is more efficient but it may lack of accuracy as emphasized in Hainaut and Akbaraly (2023).

The joint simulations of I_t and L_t allows us to compute the statistical distribution of the net asset value (NAV), defined as the difference between the total asset and the best estimate.

$$NAV_t \ := \ I_t - \mathbf{1}_{\{\beta(t)-1 < \tau_x\}} L_t \, . \tag{22}$$

The NAV is a performance measure which corresponds to the market value of future incomes earned by the insurance company. In solvency II, the indicator of risk exposure is the solvency capital requirement (SCR). This corresponds to the economic capital an insurance company must hold to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years. In solvency II, the SCR is the 0.5% percentile of the NAV distribution in one year, under the real measure $\mathbb{P}$. Interpreting the definition of the SCR in the same manner as Christiansen and Niemeyer (2014), we define the regulatory capital SCR^{reg} as follows:

$$\mathbb{P}\left(NAV_0 - NAV_1 \geq SCR^{reg}\right) \ = \ \beta \, . \tag{23}$$

where $\beta = 0.5\%$ is the confidence level. Note that we are neglecting the 1-year discount rate in this formula. The SCR defined by this way is simply an approached value of the 0.5% 1-year Value at Risk (VaR) of the NAV:

$$\mathbb{P}\left(\mathbb{E}^{\mathbb{P}}\left(NAV_1 \,|\, \mathcal{F}_0\right) - NAV_1 \geq SCR_1\right) \ = \ \beta \, . \tag{24}$$

for a confidence level of β=0.5% where the expectation is here evaluated under the real measure $\mathbb{P}$. The solvency capital calculated by this last formula is larger than SCR^{reg} for any profitable insurance company if $\mathbb{E}\left(NAV_1 \,|\, \mathcal{F}_0\right) > NAV_0$. As the solvency capital defined by equation (24) is more conservative then the one obtained with the regulator's formula, we adopt it as definition in the numerical illustrations.

[1] I_t is the total asset managed by the insurer. This is a portfolio managed as A_t but from which benefits are withdrawn.

In Solvency II, the insurance company should also ensure that its assessment of the overall solvency needs is forward-looking, including a medium term or long-term perspective as appropriate. In the Own Risk and Solvency Assessment (ORSA), the regulator proposes to evaluate the CSR as the capital needed to ensure the positivity of the NAV in 99.5% of cases. As in Hainaut et al. (2018), we instead define the SCR_t as the t-year value at risk of NAV_t:

$$\mathbb{P}\left(\mathbb{E}^{\mathbb{P}}\left(NAV_t \,|\, \mathcal{F}_{t_0}\right) - NAV_t \geq SCR_t\right) \;=\; 1-(1-\beta)^t\,, \tag{25}$$

where the confidence level is adjusted year on year. Within this approach, the yearly probability of unsolvency is β and the cumuluted probability over t years is $1-(1-\beta)^t$. We refer to Devolder and Lebègue (2016) for additional explanations about this adjustment of the confidence level.

We analyze in Section 8, the influence of the investment strategy on NAV and SCR for two types of participating contracts: an endowment and a life annuity. But before, we need to discuss the calibration of the hybrid stock-LMM.

7 Hybrid Equity-LMM calibration

Insurance contracts are valued under the risk-neutral measure; therefore, the hybrid equity-LMM should be estimated from market prices of a basket of assets and derivatives such as swaptions. We refer, for example, to Brigo and Mercurio (2007) for explanations about this procedure. Calibrating under the risk-neutral measure ensures the absence of arbitrage. Nevertheless, for long-term insurance contracts, the market may be illiquid, and calibration under the risk-neutral measure may lead to distorted results or unreliable parameter estimates. Furthermore, mortality risk is not hedgeable in financial markets, and adding (reasonable) safety margins to parameters is one way to reduce exposure to this risk. On the other hand, using the real-world measure allows you to incorporate historical data and market fundamentals, which might be valuable for long-term strategic planning or scenario analysis. Parameters under $\mathbb{P}$ are also useful for generating scenarios to compute the Solvency Capital Requirement (SCR). We refer the reader to Vedani et al. (2017) for a discussion about the choice of model parameters for insurance valuation and other pitfalls related to the incompleteness of insurance markets. Note also that without access to market data such as swaption volatilities, parameter estimates under $\mathbb{P}$, with eventual safety margins, are approximations less sensitive to short-term market fluctuations or liquidity constraints. This section details how to calibrate the hybrid equity-LMM model under $\mathbb{P}$.

We rewrite forward rates in terms of time to maturity, noted δ, before fixing and we adopt the notation

$$F(t,\delta_i) = F(t, t+\delta_i, t+\delta_i+\tau)\,,$$

to lighten developments in this section (i.e. $T_{i-1} = t+\delta_i$ and $T_i = t+\delta_i+\tau$). Under the i^{th} forward measure, the LMM equation becomes

$$d\ln\left(F(t,\delta_i)+\alpha\right) \;=\; \sigma\left(\delta_i\right)\Sigma_{i,:}\, d\boldsymbol{W}_t^{\mathbb{F}(i)}\,,$$

where $\sigma(.) : \mathbb{R}^+ \to \mathbb{R}^+$ and $\Sigma_{i,:}$ is a p-vector function of i (may be seen as a function of i, to be specified). Both are parameterized and parameters are stored in a vector $\boldsymbol{\psi}$.

Conjecture 2. *Under the real measure, we assume that forward rates are stationary:*

$$d\ln\left(F(t,\delta_i)+\alpha\right) \;\approx g_i dt + \;\; \sigma\left(\delta_i\right)\Sigma_{i,;}\, d\boldsymbol{W}_t^{\mathbb{P}} \tag{26}$$

where g_i depends only upon time to maturity. The forward stock prices, $Y_t^n = \frac{S_t}{P(t,t+\delta_n+\tau)}$, is also stationary, i.e.:

$$d\ln Y_t^n \;=\; \mu_Y dt + \sigma_S \begin{pmatrix} \Sigma_{YF} & \Sigma_{YY} \end{pmatrix} d\boldsymbol{W}_t^{\mathbb{P}} \tag{27}$$

Under these assumptions and for a small Δ, the varations of log-forward rates are:

$$\begin{aligned} & \ln\left(F(t+\Delta,\delta_i)+\alpha\right)-\ln\left(F(t,\delta_i)+\alpha\right) \\ & \approx \sigma\left(\delta_i\right)\Sigma_{i,:}\left(\boldsymbol{W}^{\mathbb{P}}_{t+\Delta}-\boldsymbol{W}^{\mathbb{P}}_{t}\right)\,, \end{aligned} \tag{28}$$

while the variation of log-forward stock price is:

$$\begin{aligned} & \ln Y^n_{t+\Delta}-\ln Y^n_t \\ & \approx \sigma_S\left(\begin{array}{cc}\Sigma_{YF} & \Sigma_{YY}\end{array}\right)\left(\boldsymbol{W}^{\mathbb{P}}_{t+\Delta}-\boldsymbol{W}^{\mathbb{P}}_{t}\right)\,. \end{aligned} \tag{29}$$

Both variations are normal random variables. We use this property to estimate the model. We assume that $p=n$ (i.e. same number of Brownian motion than forward rates). We sample the n forward rates at $m+1$ equispaced times $\{s_0,...,s_m\}$. The sampling interval is noted Δ. The times to maturity before fixing are denoted by $\{\delta_1,...,\delta_n\}$ and equal to $\delta_k=k\,\tau$. We sample forward stock prices at same dates:

$$Y^n_{s_j}=\frac{S_{s_j}}{P(s_j,s_j+\delta_n+\tau)}\qquad j=1,...,m$$

We set α exogenously (e.g. the absolute value of the minimum of libor rates over the estimation period) and calculate the first order differences of log-shifted forward rates prices (the yield):

$$x_{j,k}=\ln\left(F(s_{j+1},\delta_k)+\alpha\right)-\ln\left(F(s_j,\delta_k)+\alpha\right)$$

for $j=1\ldots m$ and $k=1...n$. The difference of log-forward stock prices are $x_{j,,n+1}=\ln Y^n_{s_{j+1}}-\ln Y^n_{s_j}$. From Eq. (28), The vector $\boldsymbol{x}_j=(x_{j,k})_{k=1,...,n+1}$ is the realization of a multivariate normal variable $\boldsymbol{X}$. The observed covariance of $\boldsymbol{x}_j$'s is denoted $\tilde{C}$. If we note $\boldsymbol{\sigma}=(\sigma(\delta_k))_{k=1,...,n}$(dim n) and $\Sigma=(\Sigma_{k,:})_{k=1,...,n}$ (dim. $n\times n$), the covariance of $\boldsymbol{X}$ is equal to $\Delta C(\boldsymbol{\psi})$ where

$$\begin{aligned} C(\boldsymbol{\psi}) &= \operatorname{diag}\left(\begin{array}{c}\boldsymbol{\sigma}\\ \sigma_S\end{array}\right)\left(\begin{array}{cc}\Sigma & 0\\ \Sigma_{YF} & \Sigma_{YY}\end{array}\right)\left(\begin{array}{cc}\Sigma^\top & \Sigma^\top_{YF}\\ 0 & \Sigma^\top_{YY}\end{array}\right)\operatorname{diag}\left(\begin{array}{c}\boldsymbol{\sigma}\\ \sigma_S\end{array}\right) \\ &= \operatorname{diag}\left(\begin{array}{c}\boldsymbol{\sigma}\\ \sigma_S\end{array}\right)\left(\begin{array}{cc}\Sigma\Sigma^\top & \Sigma\Sigma^\top_{YF}\\ \Sigma_{YF}\Sigma^\top & \Sigma_{YF}\Sigma^\top_{YF}+\Sigma_{YY}\Sigma^\top_{YY}\end{array}\right)\operatorname{diag}\left(\begin{array}{c}\boldsymbol{\sigma}\\ \sigma_S\end{array}\right) \end{aligned}$$

Parameters $\boldsymbol{\psi}$ are chosen in order to fit at best the empirical covariance matrix of $\boldsymbol{x}_j$'s, noted $\tilde{C}(\boldsymbol{x})$. We compute $\boldsymbol{\psi}$ by minimizing the mean square error between empirical and theoretical covariance matrix:

$$\boldsymbol{\psi}=\arg\min_{\boldsymbol{\psi}}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(C_{i,j}(\boldsymbol{\psi})-\tilde{C}_{i,j}(\boldsymbol{x})/\Delta\right)^2\,.$$

Based on observed standard deviations of forward rates (see Figure 1), we parameterize $\sigma(.)$ by the following function

$$\sigma\left(\delta\right)\quad=\quad\left(a\delta+d\right)e^{-b\delta}+c\,, \tag{30}$$

where $a,c,d\in\mathbb{R}^+$ and $b\in\mathbb{R}$. The $(k,j)^{th}$ element of the matrix $\Sigma\Sigma^\top$ is the correlation coefficient between $F(t,\delta_k)$ and $F(t,\delta_j)$. We adopt the following model to smooth correlations:

$$\Sigma_{k,:}\Sigma^\top_{j,:}\quad=\quad\rho_{k,j}=\rho_\infty+(1-\rho_\infty)\exp\left(|k-j|\beta(k,j)\right)\,, \tag{31}$$

where $\rho_\infty\in\mathbb{R}^+$and $\beta(.,.)$ is a function of remaining maturities before fixing of the forward rate:

$$\beta\left(k,j\right)=d_1-d_2\,\max\left(k,j\right)\,.$$

d_1 and $d_2 \in \mathbb{R}^+$. Based on observations (See Figure 2), the correlation between log-forward stock prices and forward rates is modelled by the following function

$$\left(\Sigma_{YF}\Sigma_{YF}^\top\right)_k = \rho_{SF^k} = \rho_{SF^n}\left(1 - \exp\left(-d_3 - d_4\left(n-k\right)\right)\right) . \tag{32}$$

where $d_3, d_4 \in \mathbb{R}^+$. To ensure that

$$\begin{pmatrix} \Sigma\Sigma^\top & \Sigma\Sigma_{YF}^\top \\ \Sigma_{YF}\Sigma^\top & \Sigma_{YF}\Sigma_{YF}^\top + \Sigma_{YY}\Sigma_{YY}^\top \end{pmatrix}$$

is well a correlation matrix, the following equation must be fulfilled:

$$\Sigma_{YF}\Sigma_{YF}^\top + \Sigma_{YY}\Sigma_{YY}^\top = 1 .$$

We infer from this constraint that $\Sigma_{YY} = \sqrt{1 - \Sigma_{YF}\Sigma_{YF}^\top}$. We fit the hybrid stock-LMM to 1-year forward rates computed from Belgian state bonds (maturity up to 25 years), and CAC 40 log-returns over the period 1/12/2010 to 18/3/2024 (daily observations). Parameter estimates are reported in Table 1. The average return of the stock indice is 7.56% and we round it to 8.00%, which is a reasonable estimate of the future return of this indice.

α	0.0305	d_2	0.3200
a	0.0022	ρ_∞	0.1022
b	-0.0711	σ_S	0.2999
c	0.2761	ρ_{SF}	0.3741
d	0.0001	d_3	0.6117
d_1	0.4158	d_4	0.1489
$\mu = 0.08$ (assumption)			

Table 1: Parameter estimates, hybrid stock-LMM

Figure 1 Compares empirical standard deviations of forward rates to modelled ones, with respect to time to maturity. Figure 2 presents empirical and modelled correlations between log-forward rates and log-forward stock prices, with respect to time to maturity. These graphs confirm that functions (30) and (32) are appropriate.

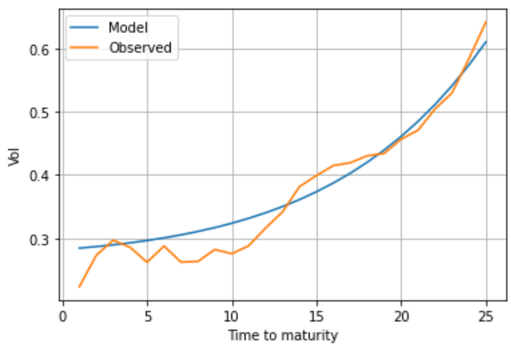

Figure 1: Empirical and modelled standard deviations of forward rates with respect to time to maturity

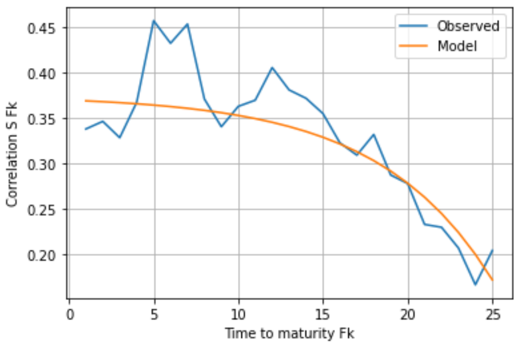

Figure 2: Observed and modelled correlations between log-forward rates and log-forward stock prices, with respect to time to maturity

8 Numerical analysis

We consider two 10–year contracts purchased by a 60-year old male individual. The first one is an endowment with minimum death and life benefits equal to 100. Benefits are paid at the end of year. The second product is a 10 year life annuity. We consider three guaranteed rates. Detailed contract features are reported in Table 2. For each product, we consider three bond investment policies: the fixed maturity, the constant duration buckets (FMB and CDB) and the cash-flow matching (CFM) strategies. We recall that the CDB strategy is the only strategy accepted by the regulator for computing best estimates in Solvency II. The motivation for such a choice is that insurance companies must simulate their investment on an ongoing basis even if liabilities cash-flows are in run-off. Maturities of bonds involved in Equations (3) and (4) defining the η_u^i's are presented in Table 3. We consider various percentage of stocks, from 0% up to 100% and the initial yield curve is the one of Belgian state bonds on the 18/3/2024. The survival probabilities are computed with a Makeham model (See Appendix B) fitted to prospective male Belgian mortality rates.

	Endowment		**Annuity**
T_g	10	T_g	10
x	60	x	60
r_g	0.0%, 1.5%, 3.0%	r_g	0.0%, 1.5%, 3.0%
d_k	100	d_k	0
l_k	l_9=100, $l_{0,\dots,8} = 0$	$l_k\,(1+r_g)^{T_k}$	$\frac{100}{a^{(0\%)}_{\overline{10}\rvert}}, \frac{100}{a^{(1.5\%)}_{\overline{10}\rvert}}, \frac{100}{a^{(3\%)}_{\overline{10}\rvert}}$

Table 2: Features of the endowment and life annuity.

Bonds portfolio					
FMB	m_1	12 years	CDB	m_1	8 years
	m_2	16 years		m_2	12 years
Stock allocation					
π from 0% to 100% by step of 5%					

Table 3: Features of the benchmark portfolios.

8.1 Valuation of best estimates

Table 5 presents the market values of endowments and life annuities, with three guarantees and various investment strategies. These best estimates are computed with parameters of Table 1 and with the closed-form expression of Proposition 4. The integral in the variance of the benchmark portfolio, Equation (11), is computed numerically with a discretization step of 0.01. In order to quantify the error induced by Conjecture 1, we evaluate the same contracts by running 1000 Monte-Carlo simulations under the Libor measure. Table 4 reports the average relative spread between Monte-Carlo and (semi-) analytical best estimates. This error varies from 0.62% to 1.74%. In most of configurations, the relative errors are around 1.00%, which is an acceptable accuracy for an ALM study. Figure 3 reveals that analytical best estimates are systematically lower than Monte-Carlo values. This observation is relevant with the assumption in Equation (1) which implicitely decreases the benchmark asset volatility. As call options embedded into contracts are proportional to this volatility, undervaluing it is at the origin of the underestimation of best estimates. In the rest of this section, liability values are exclusively computed with the approximated closed-form expression.

Average relative errors in %						
	Endowment, r_g			Annuity, r_g		
Bond strategy	0.0%	1.5%	3.0%	0.0%	1.5%	3.0%
FMB	1.173	1.486	1.512	1.04	1.037	0.871
CDB	1.292	1.656	1.743	1.077	1.104	0.953
CFM	0.94	1.067	1.151	0.669	0.674	0.623

Table 4: Average relative errors between prices

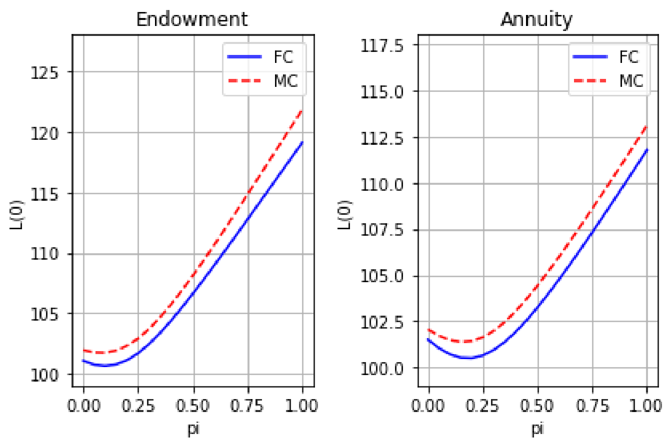

Figure 3: Best estimates with $r_g = 1.5\%$ and FMB strategy by Monte-Carlo (MC) simulations and closed-form approximation (CF)

Figure 4 shows the best estimates of the Endowment and life annuity for π that ranges from 0% to 50% and FMB, CDB and CFM strategies. This graph emphasizes the importance of the bond investment policy on the fair value of liabilities. For asset mixes with less than 20% of stocks, the lowest best estimates are obtained with a cash-flow matching allocation. Depending on the type of products, the worst strategy is either the CDB or the FMB policy. Globally, the performance depends on the mismatch between bonds and liabilities maturities. The CDB strategy, that is the only legal one in Solvency II, is then not optimal from the insurer's point of view. Figure 5 presents the best estimates of the two products for the considered guaranteed rates (r_g equal to 0%, 1.5% and 3.0%). This graph, as the previous one emphasizes the impact of diversification between stocks and bonds on liability values. Due to the limited correlation between stocks and forward rates, investing between 10% and 20% of the asset in stocks allows minimizing the best estimates.

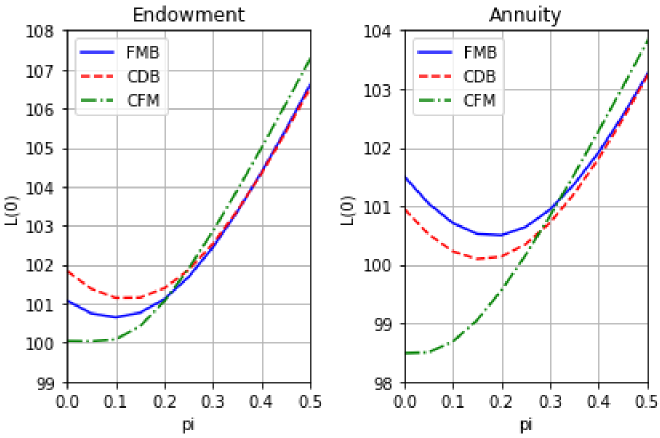

Figure 4: Best estimates with $r_g = 1.5\%$. FMB, CDB and CFM strategies

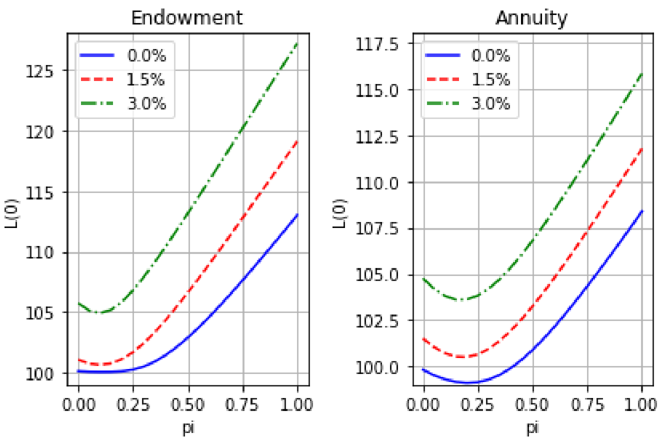

Figure 5: Best estimates for various guarantees, with FMB strategy

	Endowment			Annuity		
	FMB			FMB		
π / r_g	0.0%	1.5%	3.0%	0.0%	1.5%	3.0%
0.0	100.13	101.07	105.75	99.82	101.51	104.76
0.1	100.06	100.65	104.92	99.3	100.71	103.81
0.2	100.11	101.12	105.84	99.1	100.5	103.61
0.3	100.51	102.44	107.9	99.28	100.94	104.19
0.4	101.46	104.36	110.43	99.89	101.91	105.32
0.5	102.89	106.6	113.13	100.88	103.24	106.8
	CDB			CDB		
π / r_g	0.0%	1.5%	3.0%	0.0%	1.5%	3.0%
0.0	100.3	101.84	107.02	99.32	100.95	104.19
0.1	100.12	101.15	105.91	98.9	100.23	103.31
0.2	100.17	101.4	106.33	98.82	100.14	103.22
0.3	100.54	102.54	108.04	99.11	100.72	103.94
0.4	101.44	104.33	110.39	99.81	101.81	105.21
0.5	102.83	106.51	113.03	100.86	103.21	106.76
	CFM			CFM		
π / r_g	0.0%	1.5%	3.0%	0.0%	1.5%	3.0%
0.0	100.02	100.05	100.54	98.22	98.49	100.68
0.1	100.01	100.09	103.06	98.21	98.69	101.24
0.2	100.09	101.06	105.76	98.45	99.56	102.56
0.3	100.68	102.86	108.48	99.12	100.83	104.11
0.4	101.83	104.99	111.2	100.11	102.28	105.75
0.5	103.35	107.26	113.91	101.31	103.8	107.42

Table 5: Best estimates, endowment and life annuity

8.2 Asset-liability management indicators

Tables 6 and 7 present ALM indicators for the two contracts, with a guarantee r_g of 1.5%, for various asset allocations. We have assumed that the insurer's total asset is equal to $I_0 = 110$.

In Figures 6 and 7, we have plotted the expected 1-year NAV's with respect to the SCR, computed as the 0.5% VaR of 1-year NAV's. These results are obtained by running 10,000 simulations under $\mathbb{P}$, over a one-year time horizon. Asset sample paths are simulated by time steps of size 0.02.

We observe that the SCR widely depends on the fraction of stocks held in the portfolio. For both insurance products, it ranges from 5% up to 22% for a benchmark portfolio with 55% of stocks. We again observe a diversification effect. The SCR is respectively minimized with 10% and 15% of stocks for the endowment and life annuity. These results are aligned with market practices of European insurers that invest between 5% and 15% of their assets in equity. The bond investment policy has a limited impact on the endowment SCR. On the contrary, the cash-flow matching strategy allows reducing the SCR of the life annuity by half (from 10% to less than 5%). Nevertheless, Solvency II imposes in simulations, a reinvestment policy of bonds based on a constant duration. This option yields the highest best estimates and SCR compared to other policies and is suboptimal for the insurer.

Figures 6 and 7 reveal the importance of the bond allocation on expected NAVs. As both contracts include participation in profits above the guaranteed rate, increasing the fraction of stocks does not raise the expected NAV. We even observe a small decline for portfolios massively invested in stocks. For both products, the highest expected NAVs are obtained with CFM strategies.

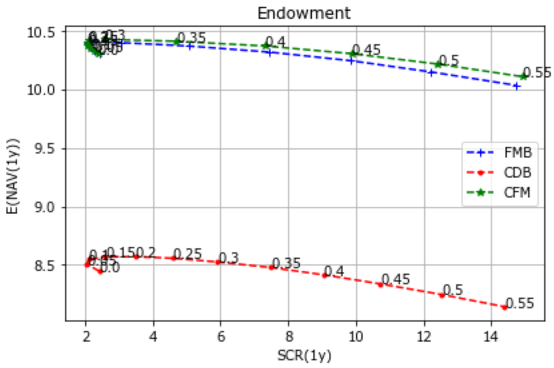

Figure 6: $\mathbb{E}^{\mathbb{P}}(NAV_1|\mathcal{F}_0)$ versus SCR_1, endowment contract. π range : 0% to 55%

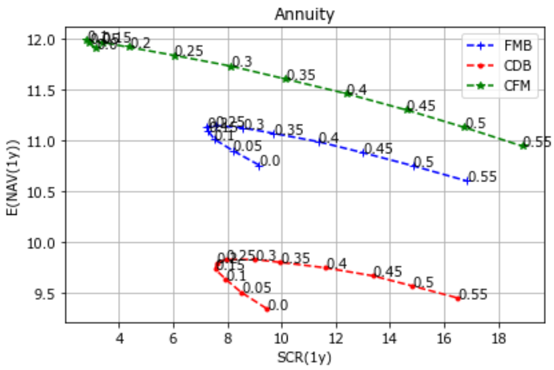

Figure 7: $\mathbb{E}^{\mathbb{P}}(NAV_1|\mathcal{F}_0)$ versus SCR_1, life annuity. π range : 0% to 55%

Figures 8 and 9 display the solvency ratios, $\frac{\mathbb{E}^{\mathbb{P}}(NAV_1|\mathcal{F}_0)}{SCR_1}$ with respect to SCR ratios, $\frac{SCR_1}{L_0}$. As the NAV is the equity of the product balance sheet, the solvency ratio must be above one to ensure the positivity of NAV in 99.5% of scenarios. This criterion is fulfilled for portfolios with less than 40% and 35% of stocks for the endowment and annuity, respectively. The SCR ratio is the percentage of equity that shareholders finance for guaranteeing the solvency. As the cost of equity is higher than the cost of other funding sources, insurers must select the asset allocation aligned with their capacity to raise and remunerate capital. The annuity and endowment respectively require a capital of up to 16% and 14% of best estimates for a portfolio with 55% of stocks. If we look to Tables 6 and 7, we observe that the highest solvency ratios and lowest SCR ratios are obtained with cash-flow matching strategies and respectively 20% and 10% of stocks for the endowment and the annuity. The CDB strategy, which is imposed in Solvency 2, significantly reduces the solvency ratio and rises the SCR ratio. This investment policy is clearly not favorable for the insurer.

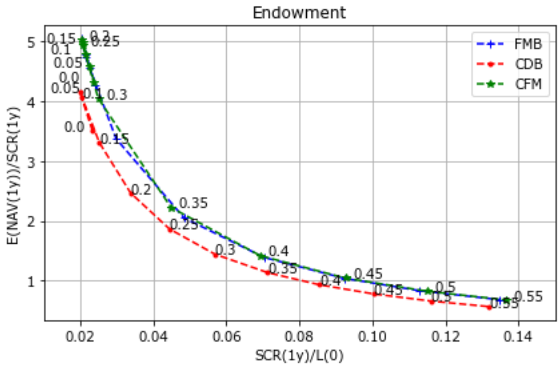

Figure 8: $\frac{\mathbb{E}^{\mathbb{P}}(NAV_1)}{SCR_1}$ versus $\frac{SCR_1}{L_0}$, endowment contract. π range : 0% to 55%

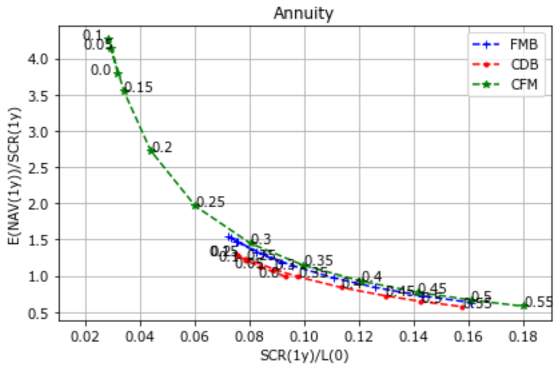

Figure 9: $\frac{\mathbb{E}^{\mathbb{P}}(NAV_{1y})}{SCR}$ versus $\frac{SCR}{L_0}$, life annuity. π range : 0% to 55%

Figure 10 presents the expected NAV's at intermediate times before expiry. We consider in this illustration a guarantee $r_g = 1.5\%$ and an asset allocation with 20% of stocks. The graphs reveal that the NAV's grow linearly and whatever the time horizon, the CFM strategy leads to the highest NAV's. Figure 11 shows the evolution of SCR's, computed with formula (24).With the CDB policy, the time to expiry of bonds in the portfolio is nearly constant whereas the time to expiry of liabilities gets shorter with time. This increasing mismatch of durations between assets and liabilities raises the exposure to interest risk. This explains why the SCRs computed with the CDB strategy are increasing functions of time. The CFM strategy is the least expensive in terms of capital. Nevertheless, we observe a slight increase of SCRs over time, mainly due to the global increase of corresponding NAVs (as NAVs are larger in absolute value, their standard deviations, and then SCRs, are also larger).

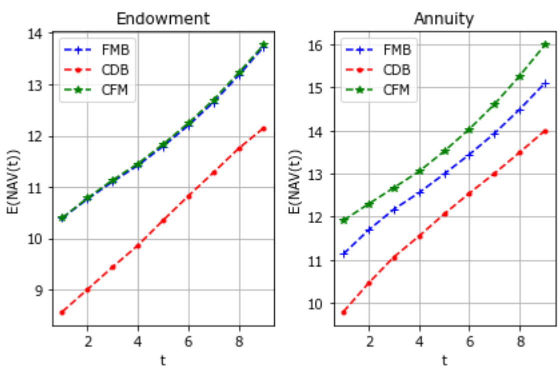

Figure 10: $\mathbb{E}^{\mathbb{P}}(NAV_t)$, endowment contract.

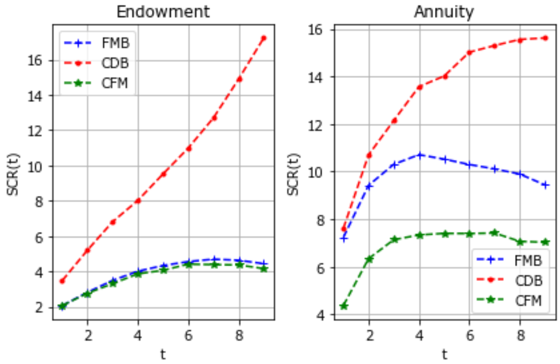

Figure 11: SCR_t, life annuity.

9 Conclusions

Based on a hybrid equity-Libor market model, we propose a framework for integrating the bond investment strategy into the valuation of participating insurances. Under the common hypothesis of stationarity, we provide a semi-analytical approximation for the best estimates. This closed-form expression, combined with Monte Carlo simulations, allows us to compute risk management indicators such as the NAV or the CSR. We also provide a method for generating asset sample paths and a calibration method from time series.

The comparison of best estimates obtained by the analytical formula and Monte Carlo simulations reveals that the closed-form approximation slightly underestimates the real values. This observation is relevant to the fact that Conjecture 1 implicitly decreases the asset volatility. As call option prices are proportional to this volatility, undervaluing it explains the underestimation of best estimates.

Our numerical results highlight the importance of the bond investment policy on best estimates. For both endowments and annuities, a cash-flow matching strategy proves to be the most efficient in reducing liability fair values. Additionally, we observe the impact of diversification between stocks and bonds. Due to the limited correlation between equity and forward rates, allocating between 10% and 20% of the asset in stocks decreases best estimates.

Without an analytical expression for liability values, we would need to conduct simulations within simulations or utilize the least squares Monte Carlo method to compute the SCR. However, the combination of primary Monte Carlo simulations under $\mathbb{P}$ with the closed-form approximation significantly reduces computation time. Furthermore, the approximation proves to be accurate enough for quantifying the influence of the investment strategy on the NAV and SCR. The ALM study clearly demonstrates that a cash-flow matching policy improves shareholder wealth creation and reduces the SCR. Additionally, we observe that investing between 10% and 20% in stocks reduces the SCR through diversification.

Appendix A

The next proposition provides the dynamics of forward rates under different forward equivalent measures:

Endowment, FMB						
π	L_0	NAV_0	$\mathbb{E}^{\mathbb{P}}\left(NAV_{1y}\right)$	SCR	$\frac{\mathbb{E}^{\mathbb{P}}(NAV_{1y})}{SCR}$	$\frac{SCR}{L_0}$
0.0	100.06	9.94	10.3	2.41	4.27	0.02
0.05	100.02	9.98	10.33	2.27	4.56	0.02
0.1	100.12	9.88	10.36	2.16	4.79	0.02
0.15	100.57	9.43	10.38	2.08	5.0	0.02
0.2	101.32	8.68	10.4	2.06	5.04	0.02
0.25	102.27	7.73	10.4	2.2	4.73	0.02
0.3	103.32	6.68	10.4	3.08	3.37	0.03
0.35	104.44	5.56	10.37	5.06	2.05	0.05
0.4	105.61	4.39	10.32	7.43	1.39	0.07
0.45	106.8	3.2	10.24	9.86	1.04	0.09
0.5	108.02	1.98	10.15	12.2	0.83	0.11
Endowment, CDB						
π	L_0	NAV_0	$\mathbb{E}^{\mathbb{P}}\left(NAV_{1y}\right)$	SCR	$\frac{\mathbb{E}^{\mathbb{P}}(NAV_{1y})}{SCR}$	$\frac{SCR}{L_0}$
0.0	103.22	6.78	8.45	2.41	3.5	0.02
0.05	102.84	7.16	8.51	2.05	4.15	0.02
0.1	102.63	7.37	8.55	2.1	4.07	0.02
0.15	102.61	7.39	8.57	2.59	3.31	0.03
0.2	102.82	7.18	8.57	3.48	2.46	0.03
0.25	103.27	6.73	8.56	4.59	1.86	0.04
0.3	103.94	6.06	8.53	5.91	1.44	0.06
0.35	104.78	5.22	8.48	7.48	1.13	0.07
0.4	105.74	4.26	8.42	9.05	0.93	0.09
0.45	106.8	3.2	8.34	10.72	0.78	0.1
0.5	107.91	2.09	8.25	12.53	0.66	0.12
Endowment, CFM						
π	L_0	NAV_0	$\mathbb{E}^{\mathbb{P}}\left(NAV_{1y}\right)$	SCR	$\frac{\mathbb{E}^{\mathbb{P}}(NAV_{1y})}{SCR}$	$\frac{SCR}{L_0}$
0.0	99.94	10.06	10.3	2.38	4.32	0.02
0.05	99.89	10.11	10.33	2.26	4.58	0.02
0.1	99.97	10.03	10.35	2.16	4.79	0.02
0.15	100.41	9.59	10.38	2.08	4.98	0.02
0.2	101.18	8.82	10.4	2.07	5.03	0.02
0.25	102.13	7.87	10.42	2.11	4.94	0.02
0.3	103.2	6.8	10.42	2.58	4.05	0.02
0.35	104.34	5.66	10.41	4.68	2.22	0.04
0.4	105.52	4.48	10.37	7.31	1.42	0.07
0.45	106.72	3.28	10.3	9.88	1.04	0.09
0.5	107.95	2.05	10.21	12.43	0.82	0.12

Table 6: Risk management indicators for the endowment

Annuity, FMB						
π	L_0	NAV_0	$\mathbb{E}^{\mathbb{P}}(NAV_{1y})$	SCR	EGN_1	$\frac{SCR}{L_0}$
0.0	99.8	10.2	10.76	9.16	1.17	0.09
0.05	99.6	10.4	10.9	8.19	1.33	0.08
0.1	99.53	10.47	11.01	7.52	1.46	0.08
0.15	99.66	10.34	11.09	7.3	1.52	0.07
0.2	99.97	10.03	11.13	7.23	1.54	0.07
0.25	100.44	9.56	11.14	7.56	1.47	0.08
0.3	101.02	8.98	11.12	8.58	1.3	0.08
0.35	101.69	8.31	11.07	9.69	1.14	0.1
0.4	102.41	7.59	10.99	11.35	0.97	0.11
0.45	103.17	6.83	10.88	12.99	0.84	0.13
0.5	103.97	6.03	10.75	14.86	0.72	0.14
Annuity, CDB						
π	L_0	NAV_0	$\mathbb{E}^{\mathbb{P}}(NAV_{1y})$	SCR	EGN_1	$\frac{SCR}{L_0}$
0.0	101.55	8.45	9.34	9.44	0.99	0.09
0.05	101.18	8.82	9.5	8.51	1.12	0.08
0.1	100.93	9.07	9.63	7.94	1.21	0.08
0.15	100.83	9.17	9.73	7.55	1.29	0.07
0.2	100.88	9.12	9.79	7.62	1.29	0.08
0.25	101.09	8.91	9.82	7.95	1.24	0.08
0.3	101.46	8.54	9.83	9.0	1.09	0.09
0.35	101.96	8.04	9.8	9.95	0.98	0.1
0.4	102.57	7.43	9.74	11.63	0.84	0.11
0.45	103.24	6.76	9.66	13.37	0.72	0.13
0.5	103.97	6.03	9.56	14.82	0.65	0.14
Annuity, CFM						
π	L_0	NAV_0	$\mathbb{E}^{\mathbb{P}}(NAV_{1y})$	SCR	EGN_1	$\frac{SCR}{L_0}$
0.0	98.56	11.44	11.91	3.14	3.79	0.03
0.05	98.55	11.45	11.96	2.9	4.13	0.03
0.1	98.74	11.26	11.98	2.81	4.26	0.03
0.15	99.15	10.85	11.97	3.37	3.55	0.03
0.2	99.71	10.29	11.92	4.37	2.72	0.04
0.25	100.37	9.63	11.84	6.04	1.96	0.06
0.3	101.09	8.91	11.73	8.12	1.44	0.08
0.35	101.84	8.16	11.6	10.14	1.14	0.1
0.4	102.62	7.38	11.46	12.39	0.92	0.12
0.45	103.42	6.58	11.3	14.6	0.77	0.14
0.5	104.22	5.78	11.13	16.76	0.66	0.16

Table 7: Risk management indicators for the endowment

Endowment						
	$\mathbb{E}^{\mathbb{P}}(NAV_t)$			SCR_t		
t	FMB	CDB	CFM	FMB	CDB	CFM
1.0	10.4	8.57	10.4	2.06	3.48	2.07
2.0	10.76	9.01	10.78	2.81	5.20	2.76
3.0	11.1	9.44	11.14	3.49	6.81	3.3
4.0	11.42	9.86	11.45	4.01	8.02	3.86
5.0	11.79	10.35	11.83	4.33	9.52	4.06
6.0	12.2	10.82	12.24	4.55	10.98	4.41
7.0	12.65	11.28	12.69	4.69	12.69	4.39
8.0	13.17	11.75	13.22	4.63	14.89	4.37
9.0	13.73	12.15	13.77	4.44	17.23	4.16
Annuity						
	$\mathbb{E}^{\mathbb{P}}(NAV_t)$			SCR_t		
t	FMB	CDB	CFM	FMB	CDB	CFM
1.0	11.13	9.79	11.92	7.23	7.62	4.37
2.0	11.69	10.46	12.3	9.42	10.7	6.34
3.0	12.17	11.06	12.67	10.31	12.14	7.14
4.0	12.56	11.55	13.05	10.71	13.57	7.34
5.0	12.99	12.06	13.52	10.52	14.01	7.41
6.0	13.44	12.55	14.04	10.3	15.02	7.4
7.0	13.94	13.01	14.61	10.11	15.29	7.42
8.0	14.49	13.5	15.27	9.9	15.55	7.07
9.0	15.1	14.0	15.99	9.44	15.61	7.03

Table 8: Expected NAV and SCR for times t from 1 to 9 years, $\pi = 20\%$.

Proposition 6. *The dynamics of F_t^j under the measure $\mathbb{F}(i)$ are the following. If $j < i$,*

$$\frac{dF_t^j}{\left(F_t^j + \alpha\right)} = -\sum_{k=j+1}^{i} \frac{\tau_k \sigma_k(t)\sigma_j(t)\rho_{j,k}\left(F_t^k + \alpha\right)}{1 + \tau_k F_t^k} dt + \sigma_j(t)\, \Sigma_{j,:}\, d\boldsymbol{W}_t^{\mathbb{F}(i)} \tag{33}$$

If $j > i$,

$$\frac{dF_t^j}{\left(F_t^j + \alpha\right)} = +\sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(t)\sigma_j(t)\rho_{j,k}\left(F_t^k + \alpha\right)}{1 + \tau_k F_t^k} dt + \sigma_j(t)\, \Sigma_{j,:}\, d\boldsymbol{W}_t^{\mathbb{F}(i)} \tag{34}$$

Appendix B

The curve of survival probabilities is described by a Makeham's model:

$$\begin{aligned} {}_tp_x^\mu &= \exp - \int_x^{x+t} \left(a^{(\mu)} + b^{(\mu)} \left(c^{(\mu)}\right)^s\right) ds \\ &= \exp(-a^{(\mu)} t) \exp\left(-\frac{b^{(\mu)}}{\ln c^{(\mu)}} \left(\left(c^{(\mu)}\right)^{x+t} - \left(c^{(\mu)}\right)^x\right)\right). \end{aligned}$$

where $a^{(\mu)}, b^{(\mu)}, c^{(\mu)} \in \mathbb{R}^+$. These parameters are obtained by least square minimization of spreads between prospective and model survival probabilities. Prospective survival probabilities are computed with a Lee-Carter model fitted to Belgian mortality rates from 1950 to 2020 for 0 to 105 years, male population. Estimated parameters are provided in Table 9.

Parameters	
$a^{(\mu)}$	1.006349e-03
$b^{(\mu)}$	2.790903e-07
$c^{(\mu)}$	1.152292

Table 9: Mortality parameters, Belgian male mortality rates, year 2020.

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References

[1] Andersen L., Andreasen J. 2000. Volatility skews and extensions of the Libor market model. Applied mathematical finance 1 (7), 1-32.

[2] Andersen L., Bortherton-Ratcliffe R., 2005. Extended Libor market models with stochastic volatility. J. of Computational finance 9 (1): 1-40.

[3] Bacinello A.R., 2001. Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. Astin Bulletin 31(2):275–298

[4] Barbarin J., Devolder P., 2005. Risk measure and fair valuation of an investment guarantee in life insurance. Insurance: Mathematics and Economics, 37 (2), 297-323

[5] Ballotta L., Haberman S., Wang N., 2006. Guarantees in with-profit and unitized with-profit life insurance contracts: Fair valuation problem in presence of the default option. The journal of risk and insurance, 73 (1), 97-121.

[6] Brace A., Gatarek D., Musiela M. 1997. The market model of interest rate dynamics. Mathematical Finance 7 (2), 127-155.

[7] Brigo D., Mercurio F. 2007. Interest rate models: theory and practice. With smile, inflation and credit. Springer-Verlag Berlin and Heidelberg GmbH & Co.

[8] Christiansen M.C., Niemeyer A. 2014. The fundamental definition of the solvency capital requirement in Solvency II. Astin Bulletin 44 (3), pp. 501 - 533.

[9] Desmettre S., Hochgerner S., Omerovic S., Thonhauser T. 2023. A mean-field extension of the LIBOR market model. arXiv:2109.10779

[10] Devolder P., Lebègue A. 2016. Risk measures versus ruin theory for the calculation of solvency capital for long-term life insurances. Dependence Modeling 4, 306-327.

[11] Gach F., Hochgerner S., Kienbacher A., Schachinger G. 2023. Mean-field Libor market model and valuation of long term guarantees. arXiv:2310.09022

[12] Gatzert N., Kling A., 2007. Analysis of participating life insurance contracts: A unification approach. The journal of risk and insurance, 74 (3), 547-570.

[13] Gatzert N., 2008. Asset management and surplus distribution strategies in life insurance: An examination with respect to risk pricing and risk measurement. Insurance: Mathematics and Economics, 42 (2), 839-849.

[14] Gerhold S. 2011. Moment explosion in the LIBOR market model, Statistics and Probability Letters 81, 560-562.

[15] Grzelak L.A., Oosterlee C.W. 2012. An equity-interest rate hybrid model with stochastic volatility and the interest rate smile. The Journal of Computational Finance, 15 (4), 1-33.

[16] Hainaut D., 2009. Dynamic asset allocation under VaR constraint, with stochastic interest rates. Annals of Operation Research 172 (1), 97–117.

[17] Hainaut D., 2010. Optimal design of profit sharing rates by FFT. Insurance: Mathematics and Economics, 46 (3), 470-478.

[18] Hainaut D., Devolder P., Pelsser A. 2018. Robust evaluation of SCR for participating life insurances under Solvency II. Insurance: Mathematics & Economics 79, 107-123.

[19] Hainaut D., Akbaraly 2023. Risk management with local least squares Monte Carlo. ASTIN Bulletin 53 (3), 489-514.

[20] Hanna V., Devolder P., 2023. Valuation of mixed life insurance contracts under stochastic correlated mortality and interest rates. European Actuarial Journal, 14, 63-98.

[21] Hull J., White A. 1996. Using Hull–White interest rate trees. Journal of Derivatives, 3 (3), 26–36.

[22] Jamshidian F. 1997. Libor and swap market models and measures. Finance and Stochastics 1(4), 293-330.

[23] Miltersen K.R., Sandmann K., Sondermann D. 1997. Closed form solutions for term structure derivatives with lognormal interest rates. J. of Finance 52(1): 409-430, 1997.

[24] Schrager D., Pelsser A. 2004. Pricing rate of return guarantees in regular premium unit linked insurance. Insurance: Mathematics & Economics 35, 369-398.

[25] Pilz K.F., Scholgl E. 2015. A hybrid commodity and interest rate market model. In Commodities, edited by M.A.H. Dempster and K. Tang. 2015 Chapman and Hall/CRC, New-York.

[26] Vedani J., El Karoui N., Loisel S., Prigent J.L. 2017. Market inconsistencies of market-consistent European life insurance economic valuations: pitfalls and practical solutions. European Actuarial Journal 7, 1-28.

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